ON THE ZEROS OF LINEAR PARTIAL FRACTIONS*

BY

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I. INTRODUCTION

1. **Problem and method.‡** The present paper owes its origin to a suggestion from Professor Walsh§ that I try to carry to a correct completion Professor Coble's analytical proof‖ of the following theorem due to the former:¶

‡ Bibliography. The following articles are cited at least twice. Each will hereafter be referred to by the letter B and a roman numeral.

BI: M. Bôcher, Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), pp. 469–484.

BII: J. L. Walsh, Comptes Rendus du Congrès International des Mathématiciens, Strasbourg, 1920, pp. 1–4.

BIII: J. L. Walsh, these Transactions, vol. 19 (1918), pp. 291–298.

BIV: J. L. Walsh, these Transactions, vol. 22 (1921), pp. 101–116.

BV: J. L. Walsh, these Transactions, vol. 24 (1922), pp. 31–69.

BVI: J. L. Walsh, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 139–141.

BVII: J. L. Walsh, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), pp. 1–13.

BVIII: A. B. Coble, Bulletin of the American Mathematical Society, vol. 27 (1921), pp. 434–437.

BIX: M. Marden, Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 363–370.

§ I wish to express my deep gratitude to Professor Walsh for his many suggestions and criticisms, and, above all, his constant encouragement in the course of this work.

‖ This theorem was proved geometrically by Walsh (see BIV). Subsequently an analytical treatment of the same problem was published by Coble (see BVIII). Although suggestive as to the manipulation of the difficult algebra of the problem, the latter article is open to the following objections. (1) It quotes Walsh's theorem incorrectly by substituting "the interior of a circle" for "the circular region." (2) It considers in the proof only the case of the regions C_i being the interiors of circles. (3) It implicitly assumes that the locus is finite whenever the regions C_i are finite. This need not be the case, as is shown in chapter IV §2 of the present paper. (4) It implicitly assumes without proof that the locus is a simply-connected region. Walsh's theorem, as stated by Coble, was later challenged by T. Nakahara (Tôhoku Mathematical Journal, vol. 23 (1924), p. 97) as if it were Walsh's own statement. Nakahara's contribution to the subject consists of a set of simple conditions for the locus of the theorem to be finite. The conditions given are, however, only sufficient, not both necessary and sufficient, as announced by Nakahara. They may be deduced more easily by the methods of chapter IV §2 of the present paper.

¶ See BIV, pp. 101–112. In Walsh's statement of the theorem, the point z is defined by means of the real constant cross ratio $(z\, z_1\, z_2\, z_3) = \lambda$. Our definition is, however, equivalent to his, provided $\lambda \neq 0, \infty$.

By a circular region is meant in the theorem the interior and circumference of a circle, the exterior and circumference of a circle, a half-plane including its boundary line, or an entire plane.

If the points z_1, z_2, z_3 vary independently and have circular regions as their respective loci, then the locus of the point z defined by the equation

(1)
$$\frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3} = 0,$$

where m_1, m_2, m_3 are real constants whose sum is zero, is also a circular region.

In the course of my attempts in this direction, I found that the methods needed in the study of this problem were sufficiently powerful for an investigation of the zeros of the more general partial fraction

$$F = \sum_{j=1}^{q+1} \frac{m_j}{z - z_j},$$

where the m_j are real constants whose sum is or is not zero. This investigation leads to a wide generalization, after the manner of Walsh, of theorems due to Gauss, Lucas, Bôcher and Walsh,* concerning the location of the zeros of the derivative of a rational function and of the zeros of the jacobian of two binary forms. For instance, its results enable one to locate approximately the positions of the zeros of the derivative of the polynomial

$$f(z) = f_1(z)f_2(z) \cdots f_k(z)$$

when all the zeros of each factor polynomial $f_i(z)$ are known to lie in or on a circle C_i. The approximation given is furthermore the best possible, in the sense that the geometrical limits involved are actually attained.

In place of merely proving the above theorem, I therefore set out to solve the following problem: *To find the locus R of the zeros of the partial fraction F when the points z_j vary independently and have circular regions C_j as their respective loci.*†

The answer to this general problem turns out to be a reasonable generalization of the above theorem. In place of a circular region as the locus of the zeros of F comes what we shall call a *p-circular domain.*‡ By a *p*-circular domain we shall mean either the entire plane or a set of simply-connected

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closed regions which contain all the ovals of a given p-circular $2p$-ic curve and which are bounded only by ovals of this $2p$-ic curve.* For example, if p were equal to two and the bicircular quartic consisted of one oval enclosed in a second, then as a bicircular domain we might have the entire plane, the interior and boundary of the outer oval, or the exterior and boundary of the inner oval, but not the ring-shaped region between the two ovals.

More specifically, we shall find the locus R to be a p-circular domain, where

$$p = q - 1 + \lim_{s \to 0} \left(\frac{n}{n+s}, \right) \text{ and } n = \sum_{j=1}^{q+1} m_j.$$

This domain, if not the entire plane, is bounded by part or all of the p-circular $2p$-ic curve with the equation†

$$\sum_{j=1}^{q+1} \frac{nm_j}{C_j} - \sum_{j=1, k=j+1}^{q+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} = 0,$$

where $C_j = 0$ is the equation of the circle bounding the circular region C_j, and where Δ_{jk} is, according to a definite rule, the square of either the external or internal common tangent of the circles C_j and C_k.

As our way to this result is rather long, it is perhaps worthwhile here to outline our method of attack and to point out its principal difficulties.

We shall obtain the above information about the locus R by successively determining certain two-dimensional envelopes.‡ If R_k denotes the locus of the zeros of F when $z_1, z_2, z_3, \cdots, z_k$ vary over their respective circular regions but the points $z_{k+1}, z_{k+2}, \cdots, z_{q+1}$ are fixed, then R_k may also be regarded as the envelope of the loci R_{k-1} when z_k describes as locus the circular region C_k. Thus, by computing successively the envelopes $R_0, R_1, \cdots, R_{q+1}$, we finally arrive at the locus R.

This method of successive envelopes entails as principal steps (1) the algebraic computation of the equations of the possible boundary curves of each locus R_k; (2) from these possible boundary curves the choice of the actual boundary curve of each locus R_k; and (3) the proof that each locus R_k is composed only of simply-connected regions.

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* We mean by a p-circular $2p$-ic curve one which is represented by an equation of the form
$$a(x^2 + y^2)^p + f(x,y) = 0$$
where $f(x, y)$ is a polynomial of degree less than $2p$. If $a \neq 0$, the curve intersects the line at infinity (in the projective plane) p times in each circular point at infinity and therefore cannot meet this line in any additional points. In other words, if $a \neq 0$, the curve consists entirely of closed ovals.

† The entire left-hand side of the equation given above lacks, of course, the factor $\prod_{j=1}^{q+1} C_j$. This factor will be omitted quite often in this paper, for the purpose of simplifying the formulas.

‡ This method is suggested in the footnote of BIV, p. 102.

As a gradual approach to the difficulties involved, we shall consider at first two special cases: in chapter II, a case (case I) where all the m_j are positive real numbers and the regions C_j are finite, and in chapter IV, case II, where the m_j are real numbers whose sum is zero, but where the regions C_j are arbitrary. The advantage of this approach is that the work once done for case I in connection with steps (1) and (3) holds almost equally for case II and that therefore in case II we are left free to overcome the difficulties of step (2) caused by the arbitrariness of the numbers m_j and the regions C_j. Furthermore, from the results of case II, we shall readily be able to infer the results in the most general case.

Having thus found the locus R, we shall try to verify our results as including those of Walsh and, finally in a closing chapter, we shall present two generalizations, the one for the zeros of the expression $F+\lambda$ where λ is a constant, real or complex, and the other for the zeros of F when the m_j are complex constants.

2. **Notation.** We now proceed to introduce a system of abridged notation which will enable us expeditiously to perform the first of the above-described steps in the determination of the loci R_k. This system is essentially the same as that used by Coble in BVIII. Like him, we employ the symbol $[C_1C_2]$ for the invariant

$$(1.21) \qquad 2(A_1A_2 + B_1B_2) - b_1a_2 - b_2a_1 \equiv a_1a_2(R_1{}^2 + R_2{}^2 - d_{12}^{2}),$$

which frequently appears in the computations. In the identity (1.21) R_i is the radius of the circle C_i, d_{12} is the distance between the centers of the circles C_1 and C_2 and the other letters are the coefficients in the equation

$$C_i \equiv a_i(x^2 + y^2) - 2A_ix - 2B_iy + b_i = 0,$$

a_i, A_i, B_i, b_i all being real numbers. If C_i is written in the form

$$C_i \equiv a_i z\bar{z} - \bar{\beta}_i z - \beta_i \bar{z} + b_i = 0,$$

where $z = x+iy$ and $\beta_i = A_i + iB_i$, then

$$[C_1C_2] = \beta_1\bar{\beta}_2 + \bar{\beta}_1\beta_2 - a_1b_2 - a_2b_1.$$

As may be easily verified, the quantity $[C_1\,C_2]$ obeys the following laws of combination:

1 $$\qquad\qquad [C_2C_1] = [C_1C_2],$$

2 $$\qquad\qquad [C_1\,kC_2] = k[C_1C_2],$$

3 $$\qquad\qquad [C_1\,C_2 + k] = [C_1C_2] - ka_1,$$

4 $$\qquad\qquad [C_1\,C_2 + C_3] = [C_1C_2] + [C_1C_3],$$

5 $$\quad [C_1 + kC_2\,C_1 + kC_2] = [C_1C_1] + 2k[C_1C_2] + k^2[C_2C_2],$$

where k is an ordinary constant.

When C_j degenerates into the null circle

$$P_j = (x - x_j)^2 + (y - y_j)^2,$$

additional laws are

6                           $$[C_j P_k] = - C_{jk},$$

7                           $$[P_k P_j] = - P_{jk} = - P_{kj},$$

where $C_{jk} = C_j(z_k)$ and $P_{jk} = P_j(z_k)$. These latter quantities have the properties that, if the point P_k is on the circle C_j, then $C_{jk} = 0$; if $P_k \equiv P_j$, then $P_{jk} = 0$; and always

$$P_{jk} + P_{jl} - P_{kl} \equiv (z_j - z_k)(\bar{z}_j - \bar{z}_l) + (\bar{z}_j - \bar{z}_k)(z_j - z_l).$$

In terms of the above notation, the radius of the circle C_j is $R_j^2 = [C_j C_j]/(2a_j^2)$ and the cosine of the angle between the two circles C_j and C_k is cos $(C_j, C_k) = [C_j C_k]/\{[C_j C_j][C_k C_k]\}^{1/2}$. As necessary and sufficient conditions we therefore have

(1) for C_j to be a null circle: $[C_j C_j] = 0$;

(2) for C_j and C_k to be orthogonal: $[C_j C_k] = 0$; and

(3) for C_j and C_k to be mutually tangent:

(1.22) $$[C_j C_k]^2 - [C_j C_j][C_k C_k] = 0.$$

The factors of the left hand side of (1.22), the invariants

$$\sigma_{jk} \equiv \frac{1}{a_j a_k}\{[C_j C_j]^{1/2}[C_k C_k]^{1/2} - [C_j C_k]\}$$

$$\equiv d_{jk}^2 - (R_j - R_k)^2,$$

and

$$\tau_{jk} \equiv - \frac{1}{a_j a_k}\{[C_j C_j]^{1/2}[C_k C_k]^{1/2} + [C_j C_k]\} \equiv d_{jk}^2 - (R_j + R_k)^2,$$

are plainly, if positive, the squares of the external and internal common tangents respectively of the circles C_j and C_k, and therefore must reduce to

$$\sigma_{jk} = \tau_{jk} = C_{kj}$$

when $C_j \equiv P_j$ and to

$$\sigma_{jk} = \tau_{jk} = P_{jk}$$

when both $C_j \equiv P_j$ and $C_k \equiv P_k$.

II. SPECIAL CASE I: POSITIVE NUMBERS m_j. FINITE CIRCULAR REGIONS C_j

1. The locus R_0. The locus R_0 (cf. definition in I § 1) consists merely of the q zero-points of the partial fraction F. Since $n \neq 0$ and the points z_j are finite, the points comprising R_0 are also finite.

The locus R_0 may be regarded as the null q-circular $2q$-ic curve

$$E_0 \equiv \left\{ \prod_1^{q+1}(z - z_j)(\bar{z} - \bar{z}_j) \right\} \left(\sum_1^{q+1} \frac{m_j}{z - z_j} \right) \left(\sum_1^{q+1} \frac{m_j}{\bar{z} - \bar{z}_j} \right) = 0.$$

To write the latter expression entirely in terms of the notation of chapter I § 2, we observe that, when $q = 1$,

$$\left(\frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} \right) \left(\frac{m_1}{\bar{z} - \bar{z}_1} + \frac{m_2}{\bar{z} - \bar{z}_2} \right) = \frac{m_1^2}{P_1} + \frac{m_2^2}{P_2}$$

$$+ \frac{m_1 m_2}{P_1 P_2} \left\{ (z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) \right\} = \frac{nm_1}{P_1} + \frac{nm_2}{P_2} - \frac{m_1 m_2 P_{12}}{P_1 P_2}.$$

This suggests what can be proved by mathematical induction; namely, that

$$E_0 \equiv \left(\prod_1^{q+1} P_j \right) \left\{ \sum_1^{q+1} \frac{nm_j}{P_j} - \sum_{\substack{j=1 \\ k=j+1}}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\}.$$

2. The locus R_1. The locus R_1, evidently a bounded point set, may be considered as the map of the region C_1 (the locus of the point z_1) by means of the transformation $z = \psi(z_1)$ defined by the equation $F = 0$. As the derivative

$$\frac{dz}{dz_1} = \frac{m_1}{(z - z_1)^2} \left[\sum_1^{q+1} \frac{m_j}{(z - z_j)^2} \right]^{-1} \not\equiv 0, \infty \ ,$$

the transformation is conformal and consequently R_1 is a set of one or more bounded two-dimensional continua.

If B_1 denotes the boundary curve of R_1 and z is any point of B_1, there must correspond to z as a point of R_1 at least (actually only) one point z_1 in the region C_1. This point z_1 must lie *on* the circle C_1. For, were z_1 inside the circle C_1, a small neighborhood of z_1 could be mapped by the function $\psi(z_1)$ upon a neighborhood of z consisting only of points of R_1. The point z could under those circumstances not be a point of B_1, contrary to the hypothesis. Consequently, when z_1 traces the circle C_1, at least part of the resulting locus of the point z is B_1.

The last fact enables us to obtain the equation of B_1 if we observe that, though representing in the variable z a q-circular $2q$-ic, the equation $E_0 = 0$ represents in the variable z_1 a null circle G_0. The null circle G_0 is, in fact,

the point z_1 corresponding to a given point z. If z is a point of B_1, by what has been proved above, G_0 lies on the circle C_1 and therefore*

$$(2.21) \qquad\qquad [E_0 C_1]_{z_1} = 0 .$$

Computed according to the rules given in I § 2, the left hand side of (2.21) is

$$E_1 \equiv C_1 \prod_2^{q+1} P_j \left\{ \sum_2^{q+1} \frac{nm_j}{P_j} - \sum_{\substack{j=2 \\ k=j+1}}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} + \prod_2^{q+1} P_j \left\{ nm_1 - \sum_2^{q+1} \frac{m_1 m_j C_{1j}}{P_j} \right\} .$$

Every point z of B_1 must therefore be a point of the curve $E_1 = 0$.

As is clear from the leading term of its equation, the curve $E_1 = 0$ is always a q-circular $2q$-ic and as such consists exclusively of closed ovals. Among these ovals we may discern two types: the first, ovals O_1 which neither contain nor are contained in any other oval of $E_1 = 0$; the second, ovals O_2 not belonging to the first type.

The facts that O_1 forms part of R_1, that R_1 is a set of finite two-dimensional regions, and that B_1 is part of the curve $E_1 = 0$ certify to the entire interior of O_1's belonging to R_1. Whether or not the part of R_1 associated with an O_2 is simply-connected, needs however further investigation. For the present, we shall assume it also to be a simply-connected region (cf. II § 6).

The locus R_1 is therefore the interior and boundary of all the ovals of the q-circular $2q$-ic curve $E_1 = 0$.

3. **The locus R_2.** Regarded as the envelope of the loci R_1, the locus R_2 consists also of a set of finite two-dimensional regions.

If z is a point of B_2, the boundary curve of R_2, to z corresponds (via the equation $F = 0$) at least one pair of points (z_1, z_2) in the regions C_1, C_2 respectively. This pair must actually lie on the circles C_1, C_2. For, were, for example, z_1 inside the circle C_1, the point z_2 could be considered fixed and the arguments of II § 2 used to contradict the hypothesis that the point z is on B_2.

In other words, if the points z_1 and z_2 trace their respective circles simultaneously, the resulting locus of the point z will include the curve B_2. That is to say, B_2 is part of the envelope of the curves B_1 or $E_1 = 0$ when the point z_2 describes as locus the circle C_2.

From this fact comes the equation of B_2 if we notice that, in the variable z_2, the equation $E_1 = 0$ represents a circle G_1. The circle G_1 corresponding to a point z on B_2 must consist of at least one point on the circle C_2 and of no points

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* The subscript z_1 means that in this operation z_1 is considered as the independent variable.

inside the circle C_2; in other words, it must be tangent to C_2. A necessary condition on z as a point of B_2 is therefore

(2.31) $$[E_1 C_2]_{z_2}^2 - [C_2 C_2]_{z_2}[E_1 E_1]_{z_2} = 0.$$

Computed by the rules of I § 2, the terms of the left hand side of this expression become

$$- [E_1 C_2] = \left(C_1 C_2 \prod_3^{q+1} P_j \right) \left\{ \sum_1^2 \frac{n m_j}{C_j} + \sum_3^{q+1} \frac{n m_j}{P_j} + \frac{m_1 m_2 [C_1 C_2]}{C_1 C_2} \right.$$

$$\left. - \sum_{j=1,k=3}^{j=2,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=3,k=j+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} ;$$

$$[E_1 E_1] = m_1{}^2 m_2{}^2 \left(\prod_3^{q+1} P_j \right)^2 [C_1 C_1].$$

Equation (2.31) accordingly factors into the two equations

$$E_2 \equiv - [E_1 C_2] - m_1 m_2 \prod_3^{q+1} P_j [C_1 C_1]^{1/2} [C_2 C_2]^{1/2} = 0,$$

$$E_2' \equiv - [E_1 C_2] + m_1 m_2 \prod_3^{q+1} P_j [C_1 C_1]^{1/2} [C_2 C_2]^{1/2} = 0,$$

at least one of which every point of B_2 must satisfy.

Both equations, having as leading term $n^2(x^2+y^2)^q$, represent q-circular $2q$-ic curves. If (ξ, η) is a point of the second, i.e. $E_2'(\xi, \eta) = 0$, then

(2.32) $$E_2(\xi, \eta) = - 2 m_1 m_2 [C_1 C_1]^{1/2} [C_2 C_2]^{1/2} \prod_3^{q+1} P_j(\xi, \eta) \leqq 0.$$

The significance of the inequality (2.32) becomes clear if we observe that the continuous function $E_2(x, y)$ grows positively infinite as the point (x, y) recedes indefinitely in any direction. The inequality (2.32) therefore expresses a sufficient condition for (ξ, η) to lie inside or on an oval of $E_2 = 0$, i.e., for every oval of $E_2' = 0$ to be nested in an oval of $E_2 = 0$.

If now we assume (cf. II § 6) that R_2 is also composed only of simply-connected parts, we obtain R_2 as the interior and boundary of all the ovals of the q-circular $2q$-ic

$$E_2 = \left(C_1 C_2 \prod_3^{q+1} P_j \right) \left\{ \sum_1^2 \frac{n m_j}{C_j} + \sum_3^{q+1} \frac{n m_j}{P_j} - \frac{m_1 m_2 \sigma_{12}}{C_1 C_2} \right.$$

$$\left. - \sum_{j=1,k=3}^{j=2,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=3,k=j+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} = 0.$$

4. **The locus R_N.** We may now venture to suppose that R_N consists of the interior and boundary of all the ovals of

$$E_N = \left(\prod_1^N C_i \prod_{N+1}^{q+1} P_i \right) \left\{ \sum_1^N \frac{nm_i}{C_i} + \sum_{N+1}^{q+1} \frac{nm_i}{P_i} - \sum_{j=1,k=i+1}^{N} \frac{m_j m_k \sigma_{jk}}{C_j C_k} \right.$$

$$\left. - \sum_{j=1,k=N+1}^{j=N,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=N+1,k=i+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} = 0.$$

On the principle of mathematical induction, let us test the correctness of this assumption by determining R_{N+1}.

Considerations similar to those of II § 3 lead to the equation

(2.41) $[E_N C_{N+1}]_{zN+1}^2 - [E_N E_N]_{zN+1}[C_{N+1} C_{N+1}]_{zN+1} = 0$

as a necessary condition that a point z lie on B_{N+1}, the boundary curve of R_{N+1}. Deciphered under the rules of I § 2, the terms in (2.41) may be written as

$$- [E_N C_{N+1}] = \prod_1^{N+1} C_i \prod_{N+2}^{q+1} P_i \left\{ \sum_1^{N+1} \frac{nm_i}{C_i} + \sum_{N+2}^{q+1} \frac{nm_i}{P_i} + \sum_1^N \frac{m_{N+1} m_i [C_{N+1} C_i]}{C_{N+1} C_i} \right.$$

$$\left. - \sum_{j=1,k=N+2}^{j=N+1,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=N+2,k=i+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} - \sum_{j=1,k=i+1}^{N} \frac{m_j m_k \sigma_{jk}}{C_j C_k} \right\},$$

$$[E_N E_N] = \left\{ \prod_1^N C_i \prod_{N+2}^{q+1} P_i m_{N+1} \sum_1^N \frac{m_i [C_i C_i]^{1/2}}{C_i} \right\}^2.$$

Every point of B_{N+1} must therefore lie on at least one of the two curves

$$E_{N+1} \equiv - [E_N C_{N+1}] - [E_N E_N]^{1/2} [C_{N+1} C_{N+1}]^{1/2} = 0$$

$$\equiv \prod_1^{N+1} C_i \prod_{N+2}^{q+1} P_i \left\{ \sum_1^{N+1} \frac{nm_i}{C_i} + \sum_{N+2}^{q+1} \frac{nm_i}{P_i} - \sum_{j=1,k=i+2}^{N} \frac{m_j m_k \sigma_{jk}}{C_j C_k} \right.$$

$$\left. - \sum_{j=1,k=N+2}^{j=N+1,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=N+2,k=i+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} = 0 ;$$

(2.42) $E_{N+1}^{(N+1)} \equiv - [E_N C_{N+1}] + [E_N E_N]^{1/2} [C_{N+1} C_{N+1}]^{1/2} = 0$

$$\equiv \prod_1^{N+1} C_i \prod_{N+2}^{q+1} P_i \left\{ \sum_1^{N+1} \frac{nm_i}{C_i} + \sum_{N+2}^{q+1} \frac{nm_i}{P_i} - \sum_{j=1,k=i+1}^{N} \frac{m_j m_k \Delta_{jk}}{C_j C_k} \right.$$

$$\left. - \sum_{j=1,k=N+2}^{j=N+1,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=N+2,k=i+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} = 0,$$

where $\Delta_{jk} = \tau_{jk}$ if i or $j = N+1$, but $\Delta_{jk} = \sigma_{jk}$ otherwise.

Let us now bring into play the fact that B_{N+1} must be independent of the

order in which the $z_i, j = 1, 2, \cdots, N+1$, have begun to vary. If, for instance, the orders of z_κ and z_{N+1} are interchanged, the condition upon z as a point of B_{N+1} must be revised so as to read that z satisfies at least one of the two equations $E_{N+1}^{(\kappa)'} = 0$ and $E_{N+1}^{(k)} = 0$. These new equations may be derived from the old (2.42) by permuting the z_κ and z_{N+1}. As the form E_{N+1} is symmetric in the subscripts $j = 1, 2, \cdots, N+1$, $E_{N+1}^{(\kappa)'} \equiv E_{N+1}$, but

$$E_{N+1}^{(k)} \equiv \prod_1^{N+1} C_j \prod_{N+2}^{q+1} P_j \left\{ \sum_1^{N+1} \frac{nm_j}{C_j} + \sum_{N+2}^{q+1} \frac{nm_j}{P_j} - \sum_{j=1,k=j+1}^{N+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} \right.$$
$$\left. - \sum_{j=1,k=N+2}^{i=N+1,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} - \sum_{j=N+2,k=j+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k} \right\} = 0,$$

where $\Delta_{ij} = \tau_{ij}$ if i or $j = \kappa$, but $\Delta_{ij} = \sigma_{ij}$ otherwise.

Let us assume that not all of curves $E_{N+1}^{(k)} = 0$ have an arc in common. If ζ is a point on the curve $E_{N+1}^{(1)} = 0$ and ν a sufficiently small arc of $E_{N+1}^{(1)} = 0$ containing ζ, there will exist at least one curve $E_{N+1}^{(\kappa)} = 0$ which does not contain ν. Let us further suppose that ζ is not on the curve $E_{N+1} = 0$. If ζ lies on B_{N+1}, it must satisfy at least one of the two equations in each of the sets

(1) $E_{N+1} = 0,\quad E_{N+1}^{(1)} = 0$;   and   (2) $E_{N+1} = 0,\quad E_{N+1}^{(\kappa)} = 0.$

Yet ζ fails to satisfy either equation of set (2). It follows therefore from (1) that every point of B_{N+1} is a point of the curve $E_{N+1} = 0$.

Up to this point we have supposed not all the curves $E_{N+1}^{(k)} = 0$ to have an arc in common. That such is the general state of affairs is evident from the distinctness in the forms $E_{N+1}^{(k)}$. In case, however, the $E_{N+1}^{(k)} = 0$ all have an arc in common, we shall use the following reasoning. We shall consider the manifold M consisting of all possible instances of special case I and its submanifold M' composed of all instances for which all the curves $E_{N+1}^{(k)} = 0$ have an arc in common. The manifold M is of dimensions $3(N+1)$, the variables being the centers and radii of the $N+1$ circles C_j. If the dimensions of M' were less than $3(N+1)$, it would be possible to draw in M a continuous arc which terminates in any chosen point of M' but does not pass through any other points of M'. It would accordingly be possible to pass to the case where the curves $E_{N+1}^{(k)} = 0$ have a common arc, continuously through cases where this is not so. As the equation E_{N+1} has coefficients depending continuously upon those of the equations $C_j = 0$, the curve B_{N+1} would consist, even in the case of the curves $E_{N+1}^{(k)} = 0$ having a common arc, only of ovals of the curve $E_{N+1} = 0$.

We have then but to prove that the dimensions of M' are less than $3(N+1)$. As the curves $E_{N+1}^{(k)} = 0$ all have a common arc, any point of this

arc has coördinates satisfying equations

$$\Gamma^{(s,t)} \equiv E_{N+1}^{(s)} - E_{N+1}^{(t)}$$

$$\equiv 2\left(\prod_1^{N+1} C_j \prod_{N+2}^{q+1} P_j\right)\left\{\frac{m_t[C_tC_t]^{1/2}}{C_t} - \frac{m_s[C_sC_s]^{1/2}}{C_s}\right\}$$

$$\left\{\sum_1^{N+1} \frac{m_j[C_jC_j]^{1/2}}{C_j} - \frac{m_s[C_sC_s]^{1/2}}{C_s} - \frac{m_t[C_tC_t]^{1/2}}{C_t}\right\} = 0,$$

where $s \neq t$ and $s, t = 1, 2, \cdots, N+1$. The common arc is thus an arc of a circle Q with various distinct equations of the form $\sum \lambda_j C_j = 0$. Equating the several expressions for the center and radius of Q, we obtain the equations

$$\frac{\sum_1^{N+1}\lambda_j\alpha_j}{\sum_1^{N+1}\lambda_j} = \frac{\sum_1^{N+1}\mu_j\alpha_j}{\sum_1^{N+1}\mu_j},$$

$$\frac{\left[\sum_1^{N+1}\lambda_j C_j, \sum_1^{N+1}\lambda_j C_j\right]}{\left(\sum_1^{N+1}\lambda_j\right)^2} = \frac{\left[\sum_1^{N+1}\mu_j C_j, \sum_1^{N+1}\mu_j C_j\right]}{\left(\sum_1^{N+1}\mu_j\right)^2}.$$

As we assumed M' to exist, these equations are compatible and define M' as a manifold of dimensions lower than $3(N+1)$.

In every case, therefore, B_{N+1} is made up only of ovals of the curve $E_{N+1} = 0$. If we assume the locus R_{N+1} to consist of simply-connected parts, it follows that R_{N+1} is the interior and boundary of all the ovals of the curve $E_{N+1} = 0$, a result which certifies to the correctness of the assumption concerning the locus R_N.

5. The locus $R \equiv R_{q+1}$. Its properties. By setting $N = q+1$ we finally conclude that *the locus R is the interior and the boundary of all the ovals of the q-circular 2q-ic curve*[*]

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[*] The following theorem also results by a limiting process from the work of the preceding sections: If $\alpha_j'(j=1, 2, \cdots, p)$ are the zeros of the partial fraction

$$\frac{m_j}{z - \alpha_j}$$

and ζ_m and ζ_M respectively the smallest and largest zeros of the partial fraction

$$\sum_{j+1}^{q+1} \frac{m_j}{z - \beta_j}$$

where $\beta_j = \Re(\alpha_j)$, then

$$\zeta_m \leqq \Re(\alpha_j') \leqq \zeta_M \qquad\qquad (j = 1, 2, \cdots, p).$$

This is Linfield's theorem for the case $k = 1$. See reference BV, p. 54, and B. Z. Linfield, these Transactions, vol. 25 (1923), p. 257.

(2.51) $$\sum_{1}^{q+1} \frac{nm_j}{C_j} \cdot - \sum_{j=1,k=j+1}^{q+1} \frac{m_j m_k \sigma_{jk}}{C_j C_k} = 0.$$

As equation (2.51) represents a circular curve, it is of interest to locate the singular foci of this curve. Proceeding in the usual way, we find the circular line through the point (u, v),

(2.52) $$y = ix + v - iu,$$

to intersect curve (2.51) in points with abscissas which satisfy the equation

$$2^q x^q \left\{ \prod_{1}^{q+1} (w - \alpha_j) \sum_{1}^{q+1} \frac{1}{w - \alpha_j} \right\} + \text{terms of lower degree} = 0,$$

where $u + iv = w$ and $a_j + ib_j = \alpha_j$. Line (2.52) will then be asymptotic to curve (2.51), and the point (u, v) a singular focus of curve (2.51) if and only if (u, v) is such that

$$\sum_{1}^{q+1} \frac{1}{w - \alpha_j} = 0.$$

The singular foci are hence the zeros of F for the $z_j = \alpha_j$; or, what amounts to the same, *the points of equilibrium in a field of force created by a particle of mass m_j at each α_j repelling according to the inverse distance law*. (See I § 1, footnotes.)*

For further geometric characterization of the singular foci when the m_j are integers, we may appeal to the theorem recently proved by Linfield that "the roots of the derivative of the rational function $f(z) = \Pi_1^n (z - z_j)^{\mu_j}$ are the multiple roots of $f(z)$ to one lower order and the foci of a curve of class $n - 1$ which touches each segment $z_j z_k$ $(j \neq k = 1, 2, \cdots, n)$ at the points dividing it in the ratio μ_j to μ_k." Our singular foci are therefore the ordinary foci of Linfield's curve.†

From the form of the equation (2.51) we can deduce two more facts about the locus R.

First, as is obvious, *any point through which pass at least three of the circles C_j is a point of curve (2.51) and therefore a point of the locus R.*

Secondly, *the common points of any two of the regions C_j belong to the locus R.*

For suppose a point P were to lie in just two of the regions C_j, for definiteness, C_1 and C_2. Entirely within the regions C_1 and C_2 let us draw respectively

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* The singular foci are therefore points of the locus R. The bicircular quartic
$$(x^2 + y^2 + 37)^2 = 64(x^2 + 25)$$
which has its singular foci at the points $(\pm 4, 0)$, points outside of its solitary oval, is an example of a p-circular $2p$-ic which, on account of the theorem, cannot serve as the boundary curve of a locus R.

† B. Z. Linfield, Bulletin of the American Mathematical Society, vol. 27 (1920), p. 17, and these Transactions, vol. 25 (1923), pp. 239–258.

new circles C_1' and C_2' which pass through the point P. The point P is now such that

$$C_1'(P) = C_2'(P) = 0$$

and $C_j(P) > 0$ for all $j \neq 1$ or 2. The expression

$$E' \equiv C_1'C_2' \prod_3^{q+1} C_j \left\{ \sum_1^2 \frac{nm_j}{C_j'} + \sum_3^{q+1} \frac{nm_j}{C_j} - \frac{m_1 m_2 \sigma_{12}'}{C_1'C_2'} \right.$$
$$\left. - \sum_{j=1,k=3}^{j=2,k=q+1} \frac{m_j m_k \sigma_{jk}'}{C_j'C_k} - \sum_{j=3,k=j+1}^{q+1} \frac{m_j m_k \sigma_{jk}'}{C_j C_k} \right\}$$

formed at P has then the value

$$E'(P) = - m_1 m_2 \sigma_{12}' C_3(P) C_4(P) \cdots C_{q+1}(P).$$

As neither one of the circles C_1' and C_2' is contained in the other, $\sigma_{12}' > 0$ and therefore $E'(P) < 0$. This means (cf. II § 3) that P lies within the ovals of the q-circular $2q$-ic curve $E' = 0$ and accordingly is a point of the locus R', the locus R in which the circles C_1' and C_2' respectively replace the circles C_1 and C_2. But, as R' is included in the R corresponding to all the original circles C_j, the point P is contained in R.

If P were assumed to be a common point of just three of the regions C_j, for example, C_1, C_2, C_3, we could draw inside each of these three regions new circles C_1', C_2', C_3' which pass through the point P. The point P would then be a point of R'', the locus R in which the new circles C' replace the old, and therefore a point of the locus R corresponding to all of the original circles C_j. Similar arguments hold for common points of four or more of the regions C_j.

6. **Simple-connectivity.** There remains for the completion of our study of special case I only to establish the fact that each locus R_k is composed of simply-connected regions. This we shall do for $k = 2$, a typical case. We draw in R_2 an arbitrary simple closed curve K and show that, without ever intersecting B_2, the curve K can be shrunk continuously to a point in R_2.

Considered as the envelope of the R_1, the locus R_2 consists of two overlapping parts: a part S_2 composed of the interior points of the R_1 corresponding to $z_2 = \alpha_2$, and a part T_2 composed of the closed areas swept out by the moving curve B_1, i.e., composed exclusively of points through which passes at least one curve B_1.

The curve K is thus composed (1) entirely of points in S_2, (2) entirely of points in T_2, or (3) partly of points in T_2 and partly of points not in T_2. If we assume the R_1 to be of simply-connected parts, there is nothing further to prove for curves K of type (1). On the same assumption, it is possible

so to deform a curve of type (3) continuously that it becomes a curve of type (2). The proof of the simple-connectivity therefore rests upon the behavior of curves K of the second type.

Let K then be a curve of the second type and w an arbitrary point on it. Through w as a point of T_2 passes at least one curve B_1, i.e., the equation

$$(2.61) \qquad\qquad E_1(w, z_2) = 0$$

has at least one solution z_2 in the region C_2 and therefore the circle G_1 given by (2.61) has at least one point in the region C_2. We shall select one of these points and label it w_2.

Let us mark the point $w = W$ on K, to which will correspond the circle $G_1 = \Lambda_1$ and the point $w_2 = W_2$ on Λ_1. Beginning at $w = W$, as w traces K continuously clockwise, the circle G_1 alters continuously its center and radius beginning with those of Λ_1. With G_1 we so shift w_2 as to cause it to trace a continuous curve Γ_2 which begins at $w_2 = W_2$. When w finally returns to W, G_1 becomes again Λ_1 but w_2 ends at a point $w_2 = W_2'$ on Λ_1, not necessarily the same as W_2. To make Γ_2 a closed curve, we add to it an arc $\lambda_2 = W_2 W_2'$ of circle Λ_1, an arc lying in the region C_2.

In the same way it is possible to associate with K a curve Γ_1 in the region C_1. We consider again any point w on K and the curve B_1 through it. As B_1 is the envelope of the curves $E_0 = 0$ when z_1 traces the circle C_1, there passes through w at least one curve $E_0 = 0$; i.e., the equation

$$(2.62) \qquad\qquad E_0(w, w_2, z_1) = 0$$

has at least one solution z_1 on the circle C_1 and therefore the null circle G_0 given by (2.62) lies on the circle C_1. We shall denote this G_0 by w_1; in particular, we shall label the points corresponding to $w = W$ and $w_2 = W_2$, and $w = W$ and $w_2 = W_2'$ by $w_1 = W_1$ and $w_1 = W_1'$ respectively.

As the point w and the corresponding point w_2 trace K and Γ_2 from W and W_2 to W and W_2', the point w_1 describes on the circle C_1 a curve Γ_1 from W_1 to W_1'. If the point w is now held fast at W and the point w_2 traces λ_2 from W_2' to W_2, the point w_1 traces an arc λ_1 of the circle C_1, which arc closes Γ_1.

So far we have been concerned with selecting, from the infinitude of pairs (z_1, z_2) leading to a point w on K, just those which together form closed curves Γ_1 and Γ_2 in the regions C_1 and C_2. The correspondence thus set up between points w of K and w_1 and w_2 of Γ_1 and Γ_2 is one-to-one except that to the point W corresponds all the points on the arcs λ_1 and λ_2. The correspondence between points w_1 of Γ_1 and w_2 of Γ_2 is one-to-one without exception.

From now on we shall consider K as generated by the point $z = w$ when w_2 and its corresponding w_1 trace out Γ_2 and Γ_1 simultaneously. By suitably deforming Γ_2 and Γ_1, we shall try to shrink K to a point in R_2.

Continuous deformations of Γ_1 and Γ_2 subject to the following conditions will preserve the essential features of the above-formed correspondences between w, w_1 and w_2: (1) the points W_1 and W_2 are to remain stationary; (2) points w_2 on λ_2 are to go into points w_2' also on λ_2, and the corresponding point w_1 into the null circle $E_0(W, w_2', w_1) = 0$; and, finally, (3) the curves Γ_1 and Γ_2 are always to be closed curves lying in the regions C_1 and C_2.

Subject to these restrictions, let us modify the shapes of Γ_1 and Γ_2 slightly, causing w_1 and w_2 to go into such points w_1' and w_2' of Γ_1' and Γ_2' that

$$\left| w_2 - w_2' \right| < \delta, \ \left| w_1 - w_1' \right| < \delta.$$

A continuous function of w_1 and w_2, w is carried into w' where

(2.63) $$\left| w - w' \right| < \epsilon,$$

and ϵ is a positive number arbitrarily small for sufficiently small δ.

Of what points the curve K', the transform of K, consists may be seen if the point w_2' and the corresponding point w_1' are allowed to trace continuously complete circuits on Γ_2' and Γ_1', beginning at the points $w_2' = W_2$ and $w_1' = W_1$. The curve K' will be a continuous curve with W and W' as terminal points, W' being another root of the equation $F = 0$ where $z_1 = W_1$ and $z_2 = W_2$.

If $W \neq W'$, the equation $F = 0$ then has $r \geqq 2$ distinct roots: $Z_1 = W$, $Z_2 = W'$, $Z_3, \cdots, Z_r$. For these a number $g(W_1, W_2)$ always exists so that

$$\left| Z_i - Z_j \right| \geqq g > 0, \ i \neq j;$$

in particular,

(2.64) $$\left| W - W' \right| \geqq g.$$

On the other hand, W' may be thought of as the point w' into which W has gone as result of the above-described deformations. From inequality (2.63) it then follows that

(2.65) $$\left| W - W' \right| < \epsilon.$$

For δ sufficiently small, we may take $\epsilon = g/2$ causing inequalities (2.64) and (2.65) to contradict one another. Consequently, $W = W'$: the new K' is a closed curve.

As total effect of our deformations on Γ_1 and Γ_2, the curve K has therefore changed into another continuous closed curve K' in R_2 which passes through the point W. If now we contract Γ_1 and Γ_2 continuously to the

points W_1 and W_2, the curve K will, without ever cutting B_2, shrink continuously to the point W, a point in R_2.

In a similar way we may prove that all the other loci R_k are composed of simply-connected parts. The curve Γ_k is chosen as a closed curve entirely in the region C_k, but the other auxiliary curves Γ_{k-p} as closed curves traced on the circles C_{k-p} by the points of tangency of certain circles G_{k-p-1}.

III. APPLICATIONS

1. The derivative of a polynomial. The results of the previous chapter will now be applied to the location of the zeros of the derivative of the polynomial:

$$f(z) = \prod_1^n (z - z_i) = \prod_1^{q+1} f_i(z),$$

where f_i is a polynomial of degree m_i. As

$$(3.1) \qquad f'(z) = f(z) \left\{ \sum_1^n \frac{1}{z - z_i} \right\},$$

the zeros of $f'(z)$ are the zeros of the partial fraction bracketed in the equation (3.1) and the multiple zeros of $f(z)$ to one lower order.*

Thus the locus of the zeros of $f'(z)$ when all the zeros of each f_i vary within and on a circle C_i will be composed of two parts: a part S due to the zeros of the partial fraction in (3.1) and a part T due to the multiple zeros of $f(z)$. The former part S has already been determined. The latter part T consists of every point of any region C_i for which $m_i \neq 1$ and the common points of any two of the regions C_i, for, at such a point, two or more points z_i can be made to coincide. But, since the common points of any two regions C_i are already included in the part S (cf. II § 5), we need not mention them again.

In short, then, *if $f_i(z)$ and $f(z) = \Pi_1^{q+1} f_i$ are polynomials of degrees m_i and n respectively, and if all the zeros of each f_i have as common locus the interior and circumference of a circle C_i, the locus of the zeros of $f'(z)$ will consist of* (a) *the whole of every region C_i for which $m_i \neq 1$ and* (b), *when $q \geq 1$, the locus R of* II § 5.

Reasoning based on the continuity of the zeros, similar to that used by Bôcher and Walsh, BI, p. 478 and BIV, p. 113, may now be employed to prove at once that there will exist exactly $m_i - 1$ zeros of $f'(z)$ in the region

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* This is essentially Gauss' theorem, Werke, vol. 3, 1816, p. 112, that the zeros of $f'(z)$ are at the multiple zeros of $f(z)$ and at the points of equilibrium in the field of force created by unit particles at each z_i repelling one another according to the inverse-distance law.

C_j and q in (b) if no two of the regions C_j overlap and if (b) does not have any points in common with any region C_j; and precisely one in each part of (b) if, furthermore, (b) breaks up into q distinct parts.

2. **Special cases.** As a generalization, the theorem of the preceding section should reduce for special values of q and special choices of the regions C_j to existing theorems on the subject. Let us see whether this is so.

When $q=0$, the locus R, according to the theorem, is merely the region C_1; that is, the zeros of $f'(z)$ lie in or on any circle enclosing the zeros of $f(z)$. This is but another way of expressing the Lucas theorem[*] that the zeros of $f'(z)$ lie within any convex polygon enclosing the zeros of $f(z)$.

When $q=1$, our theorem accords with one proved originally by Walsh (BII, p. 1): the locus consists of the circular regions C_1 (if $m_1 \neq 1$) and C_2 (if $m_2 \neq 1$), and of the interior and circumference of the circle

$$nm_1C_2 + nm_2C_1 - m_1m_2\sigma_{12} = 0,$$

with radius $r = (m_2r_1 + m_1r_2)/(m_1+m_2)$ and center $\alpha = (m_2\alpha_1 + m_1\alpha_2)/(m_1+m_2)$.

When $q=2$, our theorem again agrees with one proved by Walsh (BVI): the locus consists of the whole of every region C_j for which $m_j \neq 1$ and the interior and boundary of all the ovals of the bicircular quartic (2.51).

3. **Special cases continued.** If q is taken as arbitrary, but all the circles C_j are given a common external center of similitude, straightforward computation[†] proves the curve (2.51) to break up into the circles C_k' which have the same external center of similitude and the same common tangents as the circles C_j. The singular foci, according to their definition, are in this case the centers of the circles C_k'. All this agrees with a theorem due to Walsh (BV, p. 52).

Finally let us study the special case in which all the m_j and all the circles C_j are equal, and in which the centers of the circles C_j lie at the vertices V_j of a regular polygon of $q+1$ sides. Clearly the locus must be symmetric in the lines joining the center O of the polygon with the points V_j.

If curve (2.51) does not degenerate into a set of circles with center at O, we may choose on curve (2.51) a point P which is neither on any of the lines OV_j nor on their angular bisectors. Because of the symmetry, the circle with center at O and radius OP would intersect the curve (2.51) in $2(q+1)$ distinct points, i.e., twice more often than permissible.

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[*] F. Lucas, Comptes Rendus, 1868, p. 469, and 1888, p. 121.

[†] I have carried out the details of this computation only for the case that the external center of similitude is at the point at infinity. For $q=2$, I have also been able to show our bicircular quartic coincident with Walsh's two circles by assuming the contrary and finding that it would then intersect these two circles in a total of more than sixteen points. I hope through further study of the properties of p-circular $2p$-ic curves to be able to extend this geometrical method to the general case.

Hence the locus is composed of the circular regions C_j if $m_j \neq 1$ and of the interior and circumference of the circle with center at O and with a radius equal to the largest real root of the equation

$$(3.31) \qquad \sum_0^q \frac{3}{t_j} - \sum_{j=0,k=j+1}^q \frac{\left\{ 2h \sin \dfrac{\pi}{q+1}(j-k) \right\}^2}{t_j t_k} = 0,$$

where

$$t_j = r^2 + h^2 - a^2 - 2hr \cos \frac{2\pi j}{q+1},$$

and where a is the radius of the circle C_j and h that of the circumscribing circle of the polygon.

When $q = 2$, equation (3.31) becomes

$$(r^2 - a^2)(r^2 + h^2 - a^2) = h^2,$$

which has as largest root $r = (a^2 + ha)^{1/2}$, a number also given by Walsh (BVI, p. 140).

When $q = 3$, equation (3.31) reduces to

$$(r^2 - a^2)^3 - 2h^2 a^2 (r^2 - a^2) - h^4 a^2 = 0,$$

which, if $h = 2^{1/2}$, has as largest root a number T,

$$(3.32) \qquad (a^2 + 2a)^{1/2} < T \leq (a^2 + sa)^{1/2},$$

where $s > 2$ and is such that

$$(3.33) \qquad \frac{4}{s^3 - 4s} \leq a.$$

Equality occurs in (3.32) when and only when it takes place in (3.33).

IV. SPECIAL CASE II: NUMBERS m_j SUMMING TO ZERO. ARBITRARY CIRCULAR REGIONS C_j

1. Preliminary remarks. We now turn to the special case of real arbitrary m_j's summing to zero.

The zeros of F have in this case the striking property of being invariant under linear transformation of the plane or stereographic projection upon the sphere. If a linear transformation takes the z_j into the points z_j' the zeros of F pass into the zeros of $F(z_j')$. If a stereographic projection throws the z_j into the points w_j on the sphere, the $q-1$ zeros of F go into the $q-1$ positions of equilibrium on the sphere in the field of force due to particles

of mass m_j at the w_j repelling according to the inverse-distance law. (Cf. BI, p. 469 ff.) If, then, we project the regions comprising the locus R upon the sphere, the arguments of II § 6 will serve to show that the images of these regions upon the sphere are simply-connected.

From the formal work of the second chapter, we infer at once that, when not the entire plane, the locus R in case II is bounded by ovals of the curve

$$(4.1) \qquad \sum_{j=1,\,k=j+1}^{q+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} = 0.$$

A new choice of Δ_{jk} between σ_{jk} and τ_{jk} has however to be made in special case II for several reasons. No longer are the previous choices generally valid, because the m_j and the leading coefficient of the left-hand side of equation (4.1) may now be either positive or negative, and because the regions C_j and the locus R may be finite or not finite.

To pass gently from chapter two, let us first make the new choice for the subcase in which both all the regions C_j and the locus R are finite. From this subcase we shall then proceed to the general situation by causing the regions C_j of the subcase to expand continuously until they approach the C_j of the general case.

But, before we may discuss this subcase, we need to obtain criteria for determining whether or not a given instance of finite regions C_j will yield a finite locus R.

2. **Criteria for a finite R.** That finiteness in the regions C_j does not ensure a finite locus R is evident from the example

$$(4.21) \qquad \frac{1}{z-1} + \frac{1}{z+1} - \frac{2}{z}$$

which has $z = \infty$ as its only zero, the regions C_j being the points $z = 0$ and $z = \pm 1$. (Contrast with BVIII.)

To obtain a condition for a finite R, let us express the equation $F = 0$ in the form

$$z^{q-1} \left\{ \sum_{1}^{q+1} m_j z_j \right\} + \text{terms of lower degree} = 0.$$

It is thus necessary and sufficient for $z = \infty$ to be in R that there exist in each region C_j a point z_j making

$$(4.22) \qquad \sum_{1}^{q+1} m_j z_j = 0.$$

In order to give this condition a more practical expression, let us determine the locus L_{12} of the point z_{q+1} defined by (4.22) when z_1 and z_2 vary respectively

over the regions C_1 and C_2. Applying the method of I § 2, we find here R_0 to be the null circle

$$e_0 = -\left(\sum_1^{q+1} m_j z_j\right)\left(\sum_1^{q+1} m_j \bar{z}_j\right) = \sum_{j=1,k=j+1}^{q+1} m_j m_k P_{jk} = 0,$$

and R_2 or L_{12} finally to be the interior and circumference of the circle

$$D_{12} = m_1 m_2 \Delta_{12} + \sum_{j=1,k=3}^{j=2,k=q+1} m_j m_k C_{jk} + \sum_{j=3,k=j+1}^{q+1} m_j m_k P_{jk} = 0$$

with center at

$$-\frac{1}{m_{q+1}}\left\{\sum_1^2 m_j \alpha_j + \sum_3^q m_j z_j\right\}.$$

As the radius of D_{12} is

$$\left|\frac{m_1 r_1 + m_2 r_2}{m_{q+1}}\right| \quad \text{if} \quad \Delta_{12} = \sigma_{12},$$

and

$$\left|\frac{m_1 r_1 - m_2 r_2}{m_{q+1}}\right| \quad \text{if} \quad \Delta_{12} = \tau_{12},$$

the quantity Δ_{12} must be chosen equal to σ_{12} if $m_1 m_2 > 0$, and to τ_{12} if $m_1 m_2 < 0$.

When and only when this locus L_{12} overlaps the circular region C_{q+1} will condition (4.22) be fulfilled and the point $z = \infty$ belong to the locus R. In other words, for the locus R to be finite, it is necessary and sufficient that every point z_{q+1} of the region C_{q+1} satisfy the inequality $D_{12}(z_{q+1}) < 0$.

3. **The finite subcase.** Let us now determine the quantity Δ_{jk} in equation (4.1) for the subcase of finite regions C_j and a finite locus R.

From the work of the second chapter, it is plain that to determine for instance Δ_{12}, it is only necessary to specialize in (4.1) all the C_j except C_1 and C_2 to be null circles, and to choose between the two equations

$$E_2 = C_1 C_2 \prod_3^{q+1} P_j \left\{\frac{m_1 m_2 \sigma_{12}}{C_1 C_2} + \sum_{j=1,k=3}^{j=2,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} + \sum_{j=3,k=j+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k}\right\} = 0,$$

$$E_2' = C_1 C_2 \prod_3^{q+1} P_j \left\{\frac{m_1 m_2 \tau_{12}}{C_1 C_2} + \sum_{j=1,k=3}^{j=2,k=q+1} \frac{m_j m_k C_{jk}}{C_j P_k} + \sum_{j=3,k=j+1}^{q+1} \frac{m_j m_k P_{jk}}{P_j P_k}\right\} = 0.$$

The leading coefficient of the first is precisely D_{12} with $\Delta_{12} = \sigma_{12}$ and that of the second, D_{12} with $\Delta_{12} = \tau_{12}$ (cf. IV § 2).

If $m_1 m_2 > 0$ and ζ is any point on $E_2' = 0$, i.e., $E_2'(\zeta) = 0$, then

(4.31) $$E_2(\zeta) = 2 m_1 m_2 [C_1 C_1]^{1/2} [C_2 C_2]^{1/2} \prod_3^{q+1} P_j(\zeta) \geqq 0.$$

Since $D_{12} < 0$, inequality (4.31) signifies that all the ovals of the curve $E_2' = 0$ are contained in those of the curve $E_2 = 0$. Similarly, if $m_1 m_2 < 0$, all the ovals of the curve $E_2 = 0$ are enclosed in ovals of the curve $E_2' = 0$.

The choice for Δ_{12} is typical. The locus R consists of the interior and boundary of all the ovals of the $(q-1)$-circular $2(q-1)$-ic curve

$$(4.32) \qquad \sum_{j=1, k=j+1}^{q+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} = 0,$$

where $\Delta_{jk} = \sigma_{jk}$ if $m_j m_k > 0$, and $\Delta_{jk} = \tau_{jk}$ if $m_j m_k < 0$.

4. The general case. For convenience in our treatment of the unrestricted instance of special case II, let us introduce the notion of the "radius of a circular region C." Considering that direction positive along radii vectores which points into the region C, we define as the radius of the region C ($\tilde{r}$ where r is the radius of circle C) the algebraic distance from the circumference to the center of C. Thus if the region C were the exterior of the circle C, we should find $\tilde{r} < 0$.

On each circle C_j let us now choose a point z_j^* such that

$$(4.41) \qquad \sum_1^{q+1} m_j z_j^* \neq 0,$$

and construct within the region C_j a circle γ_j, of radius ω_j, tangent to the circle C_j at z_j^*. Because of condition (4.22) the locus of the zeros of F when the z_j vary over the regions γ_j (the interior and circumference of the circle γ_j) will be finite when the radii ω_j are sufficiently small. It will be in fact bounded by curve (4.32) where the C_j are replaced by the γ_j.

If we allow the ω_j gradually to increase, the coefficients of (4.32) will change continuously. If $\tilde{r}_p < 0$, the stereographic projection of the circle γ_p upon the sphere must experience two stages in order to coincide with the circle C_p: first the circle γ_p must expand until it passes through the north pole of the sphere, i.e., $\tilde{\omega}_p$ must grow positively infinite; then it must continue to expand, thereafter the region γ_p enclosing the north pole. That is to say, the quantity $\tilde{\omega}_p$ must increase from $-\infty$ to $\tilde{r}_p$.

In equation (4.32), accordingly, the forms only of

$$\sigma_{jk} = d_{jk}^2 - (r_j - r_k)^2$$

and

$$\tau_{jk} = d_{jk}^2 - (r_j + r_k)^2$$

are affected by the continuous expansions of the regions γ_j. If $\tilde{r}_j \tilde{r}_k < 0$, σ_{jk} and τ_{jk} are interchanged; otherwise, they too retain their original forms.

In other words, provided, as we have implicitly assumed, the locus R

is not the entire plane, it consists of a set of simply-connected regions bounded by ovals of the $(q-1)$-circular $2(q-1)$-ic curve

$$(4.42) \qquad \sum_{j=1,\,k=j+1}^{q+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} = 0,$$

where $\Delta_{jk} = \sigma_{jk}$ if $(m_j \tilde{r}_j)(m_k \tilde{r}_k) > 0$ and $\Delta_{jk} = \tau_{jk}$ if $(m_j \tilde{r}_j)(m_k \tilde{r}_k) < 0$. In short, *the locus R is always a $(q-1)$-circular domain.*

To be noticed also is the type of curve which bounds the locus R. Since the quantities Δ_{jk} are invariants and the $C_j(x, y)$ covariants of the forms $C_j(x_j, y_j)$, the curve (4.42) is also a covariant of these forms. This is to be expected from the remarks of IV § 1.

The locus R, just determined, may or may not include the point at infinity. Conditions that it should contain this point have already been found (IV § 2) in the case that all the $\tilde{r}_j$ are positive. It is now desirable that we seek similar conditions for the case of unrestricted $\tilde{r}_j$.

Suppose, first, that $-\infty \leqq \tilde{r}_1 < 0$ but $0 < \tilde{r}_2 < \infty$. By condition (4.22), when $z_1 = \infty$, $z_{q+1} = \infty$ and therefore the point $z = \infty$ will be part of the locus L_{12}. (Cf. IV § 2.) As $\bar{\omega}_1$ increases from $-\infty$, the quantities σ_{12} and τ_{12} are interchanged. In any case, for large values of $-\bar{\omega}_1$, the radius of circle D_{12} will be

$$- (1/|m_{q+1}|) \, (|m_2| \bar{\omega}_2 + |m_1| \bar{\omega}_1).$$

If, then, $\bar{\omega}_1$ continues to increase to the value $-|m_2/m_1| \bar{\omega}_2$, D_{12} becomes a null circle and L_{12} the entire plane. Briefly, if $r_1 \leqq |m_2/m_1| r_2$, the locus L_{12} is the entire plane; if $r_1 > |m_2/m_1| r_2$, the locus L_{12} is the exterior of the circle D_{12} where $\Delta_{12} = \tau_{12}$ if $m_1 m_2 > 0$ and $\Delta_{12} = \sigma_{12}$ if $m_1 m_2 < 0$.

If both $\tilde{r}_1 < 0$ and $\tilde{r}_2 < 0$, the locus L_{12} is always the entire plane. For, in allowing γ_1 and γ_2 to expand, we can choose $\bar{\omega}_1$ as negative and $\bar{\omega}_2$ as positive and so large that

$$\bar{\omega}_1 \geqq - |m_2/m_1| \bar{\omega}_2.$$

In other words, if $\tilde{r}_j < 0$ for any two j, the locus R always contains the point at infinity.*

---

* The point z_{q+1} defined by the equation (4.22) is the center of gravity G of the other points z_j. When the first q z_j vary over their regions C_j, the locus of G will be, according to our methods,
(a) the entire plane, if, for at least two k, $\tilde{r}_k < 0$, or if, for at least one j and one k, $\tilde{r}_j < 0$ and

$$|\tilde{r}_j| \leqq |m_k/m_j| \, |\tilde{r}_k| \, ;$$

(b) the interior and circumference of the circle

$$H \equiv \sum_{\substack{i=1 \\ j=i+1}}^{q} m_i m_j \Delta_{ij} + m_{q+1} \sum_{1}^{q} m_j C_{j\,q+1} = 0$$

if $\tilde{r}_k > 0$ all k; or,
(c) the exterior and circumference of the circle H if $\tilde{r}_k < 0$ for just one k.

The quantity Δ_{ij} is the same in H as it is for the curve (4.42). For another derivation of (b), see BV, p. 60.

When $q=2$, an alternative method for deciding whether $z=\infty$ is part of the locus R may be obtained directly. Equation (4.42) represents a circle which, formed for the γ_j, is

(4.421) $\Gamma = m_1m_2\Delta_{12}\gamma_3 + m_1m_3\Delta_{13}\gamma_2 + m_2m_3\Delta_{23}\gamma_1 = 0,$

with radius and center respectively

(4.43) $\omega = \left| \dfrac{\nu}{\lambda} \right| \quad \text{and} \quad \alpha = \dfrac{\mu}{\lambda},$

where

$$\lambda = m_1m_2\Delta_{12} + m_1m_3\Delta_{13} + m_2m_3\Delta_{23},$$

$$\mu = m_1m_2\Delta_{12}\alpha_3 + m_1m_3\Delta_{13}\alpha_2 + m_2m_3\Delta_{23}\alpha_1,$$

$$\nu = m_1m_2\Delta_{12}\rho_3 + m_1m_3\Delta_{13}\rho_2 + m_2m_3\Delta_{23}\rho_1.$$

Since, when the $\omega_j=0$, all j, $\Delta_{jk}>0$, and

$$\lambda = -\left(\sum_1^3 m_j z_j^* \right) \left(\sum_1^3 m_j \bar{z}_j^* \right) < 0,$$

we always have $\bar{\omega} = -\nu/\lambda$. Consequently, the locus R is finite or not according as $\bar{\omega}>0$ or $\bar{\omega}<0$.

5. The entire plane as locus. To complete the study of the general case, we need yet to inquire into the circumstances under which R is the entire plane.

As the regions γ_j expand, one or more of the ovals of (4.32) may conceivably expand and sweep out the entire plane; or, after R has acquired the point $z=\infty$, the oval bounding the part of R containing $z=\infty$ may shrink to a point and its exterior thus become the entire plane. When $q=2$, these descriptive conditions are easily expressible in terms of ω and α of (4.43).

The locus R may also become the entire plane discontinuously. If all the regions C_j have a point in common, the instant the regions γ_j are sufficiently large all to include this common point does the equation $F=0$ become the identity

$$\left(\frac{1}{z-z_1} \right) \sum_1^{q+1} m_j = 0,$$

and therefore locus R the entire plane. An example of this sort has been called to my attention by Professor Walsh. (See also B VII, p. 12.) If $q=2$, $m_1=m_2=1$, the regions C_1 and C_2 are the interior of the circle $C_1 \equiv C_2$, and the region C_3 is merely the point at infinity, then R is the locus of the zeros of

$$F = \frac{1}{z - z_1} + \frac{1}{z - z_2}$$

or the locus of the point $z = \frac{1}{2}(z_1 + z_2)$. Obviously R is the interior of the circle $C = C_1$. If the region C_1 is suitably allowed to expand, the region C remains coincident with it, until the region C_1 becomes a half-plane when, because all three regions C_j have a common point, the region C suddenly becomes the entire plane.

V. APPLICATIONS

1. A cross-ratio theorem. We now return to the theorem which was the starting point of this paper (cf. I §1), the theorem that "if the points z_1, z_2, z_3 vary independently and have circular regions as their respective loci, then the locus of the point z defined by the equation

$$(5.1) \qquad \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3} = 0,$$

where m_1, m_2, m_3 are real constants whose sum is zero, is also a circular region."

By specializing throughout the preceding chapter the number q to be two, we obtain at once (1) an analytical proof of this theorem; (2) conditions that the locus contain the point at infinity; (3) conditions that the locus be the entire plane; and (4) the equation of the circle bounding the locus when the locus is not the entire plane. The latter equation is (cf. equation (4.42))

$$m_1 m_2 \Delta_{12} C_3 + m_1 m_3 \Delta_{13} C_2 + m_2 m_3 \Delta_{23} C_1 = 0,$$

where $\Delta_{jk} = \sigma_{jk}$ if $(m_j \tilde{r}_j)(m_k \tilde{r}_k) > 0$ and $\Delta_{jk} = \tau_{jk}$ if $(m_j \tilde{r}_j)(m_k \tilde{r}_k) < 0$.

As we remarked in the fifth footnote to chapter I, the above theorem may also be expressed in terms of a real constant cross ratio, instead of in terms of the partial fraction (5.1).

2. The jacobian of two binary forms. A second immediate application of the results of the preceding chapter is to the location of the zeros of the jacobian of the two binary forms

$$f_1 = \prod_{1}^{p_1}(uv_j - vu_j) \quad \text{and} \quad f_2 = \prod_{p_1+1}^{p_1+p_2}(uv_j - vu_j).$$

By Euler's law for homogeneous functions, this jacobian J_{12} may be written as

$$(5.21) \qquad J_{12} = \frac{f_1 f_2}{v^2} \left\{ \sum_{1}^{p_1} \frac{p_2 v_j v}{uv_j - vu_j} + \sum_{p_1+1}^{p_1+p_2} \frac{(-p_1)v_j v}{uv_j - vu_j} \right\}.$$

The zeros of J_{12} are thus either multiple zeros of one lower order of the pro-

duct $f_1 f_2$, or zeros of the bracket in (5.21). (Cf. BI, p. 476.) On introduction of the non-homogeneous coördinates $z = u/v$, this bracket becomes the partial fraction

$$F = \sum_{1}^{p_1} \frac{p_2}{z - z_j} + \sum_{p_1+1}^{p_1+p_2} \frac{(-p_1)}{z - z_j} .$$

As in III §1, we now obtain the following theorem:

If f_{ij}, $f_1 = \prod_1^s f_{1j}$, and $f_2 = \prod_{s+1}^{q+1} f_{2j}$ are binary forms of degrees n_{ij}, $1/p_1$, $-1/p_2$ respectively, and if all the zeros of each f_{ij} have as common locus a circular region C_j, the locus of the zeros of J_{12} will consist of (a) *the whole of every region C_j for which $n_{ij} \neq 1$, and* (b), *when $q \geq 2$, the $(q-1)$-circular domain specified in IV §4 and §5, where $m_j = p_i n_{ij}$.*

Reasoning similar to that referred to in III §1 will also establish the facts that there will exist in each region C_j exactly $n_{ij} - 1$ zeros of J_{12} and $q - 1$ in (b) if no two of the regions C_j have any point in common and if, in addition, (b) does not overlap any region C_j; and precisely one zero in each part of (b) if furthermore (b) falls into $q - 1$ distinct parts.

When $q = 1$, the theorems of this section become identical with those of Bôcher (BI, p. 477) that, if all the zeros of f_1 and f_2 lie respectively in the circular regions C_1 and C_2, there are no zeros of J_{12} outside of these two regions, and that, if the regions C_1 and C_2 do not overlap, there are exactly $p_1 - 1$ zeros of J_{12} in the first and $p_2 - 1$ zeros of J_{12} in the second region.

When $q = 2$, the theorems of this section reduce to those due to Walsh (BIV, p. 112) that, if the circular regions C_1, C_2, C_3 are the respective loci of m_1 roots of f_1, the remaining m_2 roots of f_1, and all the m_3 roots of f_2, then the locus of zeros of J_{12} will consist of the whole of each circular region C_j for which $m_j \neq 1$ and the locus described in V §1.

Similar theorems hold true for the covariant

$$\Phi = \frac{1}{k}\left(\prod_1^k f_j\right) \sum_{i,j=1}^{k} \frac{m_i}{p_i} \frac{J_{ij}}{f_i f_j}$$

of the forms

$$f_j = \prod_{P_{j-1}+1}^{P_j} (uv_i - vu_i)$$

where $P_0 = 0$, and $P_j - P_{j-1} = p_j$, the degree of f_j, and $\sum_1^k m_i p_i = 0$. For, we may write

$$\Phi = \frac{1}{v^2}\left(\prod_1^k f_j\right) \sum_1^k m_i \sum_{P_{j-1}+1}^{P_j} \frac{vv_j}{uv_j - vu_j} .$$

(See BI, p. 467.)

VI. The most general case

1. The locus R. Having obtained the locus R in the special case II (the case of real numbers m_j summing to zero but of arbitrary circular regions C_j) we are now in a position to study the locus R in the case of unrestricted numbers m_j and of unrestricted circular regions C_j. For, as we shall presently see, the results in the most general case can be deduced from those of special case II.

Let us consider the partial fraction

$$(6.11) \qquad F = \sum_{1}^{q+1} \frac{m_j}{z - z_j} + \frac{m_{q+2}}{z - z_{q+2}},$$

where $n + m_{q+1} = 0$ and $n = \sum_{1}^{q+1} m_j$. This fraction is of the type considered in the special case II, but, if the point z_{q+2} is required to recede to infinity, the term in the subscript $q+2$ disappears from expression (6.11) and the fraction becomes one in which the sum of the m_j's is not necessarily zero.

In the particular instance of case II in which $q+2$ circular regions C_j appear and the region C_{q+2} consists merely of the point z_{q+2}, the locus R is bounded by the curve (cf. equation (4.42))

$$(6.12) \qquad \sum_{j=1,\,k=j+1}^{q+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} + \sum_{1}^{q+1} \frac{m_{q+2} m_j C_{j\,q+2}}{P_{q+2} C_j} = 0 .$$

Let us now cause the point z_{q+2} to move off to infinity. The above-considered locus R will then become the locus R of the unrestricted case. Since

$$\frac{C_{j\,q+2}}{P_{q+2}} = \frac{x_{q+2}^2 + y_{q+2}^2 - 2a_j x_{q+2} - 2b_j y_{q+2} + c_j}{x_{q+2}^2 + y_{q+2}^2 - 2x x_{q+2} - 2y y_{q+2} + x^2 + y^2}$$

and

$$\lim_{|z_{q+2}| \to \infty} \left(\frac{C_{j\,q+2}}{P_{q+2}} \right) = 1,$$

the locus R of the general case will therefore be bounded by the curve

$$(6.13) \qquad \sum_{j=1,\,k=j+1}^{q+1} \frac{m_j m_k \Delta_{jk}}{C_j C_k} - \sum_{1}^{q+1} \frac{n m_j}{C_j} = 0,$$

where $\Delta_{jk} = \sigma_{jk}$ if $(m_j \tilde{r}_j)(m_k \tilde{r}_k) > 0$ and $\Delta_{jk} = \tau_{jk}$ if $(m_k \tilde{r}_k)(m_k \tilde{r}_k) < 0$.

Thus we obtain a curve which clearly includes as special cases the curves (2.51) and (4.42) and which has, if $n \neq 0$, the same singular foci as curve (2.51).

In the same manner, we can carry over from the special case II all of

the other important properties of the locus R. We see thereby that *in every case the locus R is a p-circular domain, where*

$$p = q - 1 + \lim_{\nu \to 0} \left(\frac{n}{n + \nu} \right).$$

2. Application to the derivative of a rational function. As a final application let us consider the location of the zeros of the derivative of the rational function

$$f(z) = \frac{\prod_1^u (z - z_j)}{\prod_{u+1}^{u+v} (z - z_j)},$$

where the numerator z_j are distinct from the denominator ones. Since

$$(6.21) \qquad f'(z) = f(z) \left\{ \sum_1^u \frac{1}{z - z_j} - \sum_{u+1}^{u+v} \frac{1}{z - z_j} \right\},$$

the zeros of $f'(z)$ are the multiple zeros of $f(z)$ to one lower order, the point at infinity if $v - u > 0$ and the zeros of the partial fraction in the brace. We may accordingly state the following theorem, which, when $q = 1$ or 2, reduces to those proved by Walsh in BIV, pp. 114–115:

If $f_{1j}(z)$, $f_{2i}(z)$, $f_1 = \Pi_1^s f_{1j}$, and $f_2 = \Pi_{s+1}^{q+1} f_{2i}$ are polynomials of degrees m_j, $-m_i$, u and v respectively; if f_1 and f_2 have no zeros in common, and if all the zeros of each polynomial f_{ik} have as common locus a circular region C_k, then the locus of the zeros of the derivative of the rational function $f(z) = f_1/f_2$ will be
   (a) *the whole of each region C_j for which $m_j \neq 1$ and $j = 1, 2, 3, \cdots, s$;*
   (b) *the point at infinity if $v - u > 0$, and*
   (c) *if $q \geq 1$, the p-circular domain specified in the preceding section, VI § 1.*

VII. Generalizations

1. Complex m_j. The methods of the preceding chapters permit us also to study the locus R of the zeros of the partial fraction F when the numbers m_j are complex.* We give here only the results in two special cases: case (a), where $q = 1$, $m_1 + m_2 \neq 0$, and the circular regions C_j are finite; and case (b), where $q = 2$, $m_1 + m_2 + m_3 = 0$, and both the regions C_j and the locus R are finite.

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* Cf. BVII for a geometrical discussion of cases of complex m_j, and BIX for a study of simple zero-free regions of the fraction F in the case of complex m_j; also Walsh, these Transactions, vol. 24 (1922), pp. 163–180; p. 169.

We shall in the statement of these results use the following additional notation:

$$\mu_{jk} = \frac{1}{2}(\bar{m}_j m_k + \bar{m}_k m_j) \; ; \quad \nu_{jk} = \frac{i}{2}(\bar{m}_j m_k - m_j \bar{m}_k) \; ;$$

$$l_{z_j z_k} = i(\bar{z}_j z_k - \bar{z}_k z_j) \; ;$$

$$L_{jk} = i\{(\bar{z} - \bar{\alpha}_j)(z - \alpha_k) - (z - \alpha_j)(\bar{z} - \bar{\alpha}_k)\} .$$

In subcase (a), the locus R is the interior and circumference of the circle

$$(|m_1|^2 + \mu_{12})C_2 + (|m_2|^2 + \mu_{12})C_1$$
$$- \{\mu_{12}\sigma_{12} + 2(|m_1 m_2| - \mu_{12})r_1 r_2 - \nu_{12}L_{12}\} = 0,$$

with center at the point

$$\frac{1}{|m_1 + m_2|^2}\{(|m_1|^2 + \mu_{12})\alpha_2 + (|m_2|^2 + \mu_{12})\alpha_1\}$$

and with a radius equal to

$$\frac{|m_1| r_2 + |m_2| r_1}{|m_1 + m_2|} .$$

In subcase (b), the subcase of V § 1 for complex λ, the locus R will be finite provided that the region C_3 does not overlap the interior or circumference of the circle

$$E = \mu_{23}C_{23} + \mu_{13}C_{13} + \mu_{12}r_{12} + 2(|m_1| |m_2| + \mu_{12})r_1 r_2$$
$$- \nu_{23}(l_{\alpha_1 \alpha_2} + l_{\alpha_2 \alpha_3} + l_{z_3 \alpha_1}) = 0,$$

i.e., provided that. for all points z_3 in the region C_3,

$$E(z_3) < 0.$$

The locus R then will consist of the interior and circumference of the circle

$$H = \{\mu_{12}r_{12} + 2(|m_1| |m_2| + \mu_{12})r_1 r_2 + \nu_{12}L_{12}\}C_3$$
$$+ \{\mu_{31}r_{31} + 2(|m_3| |m_1| + \mu_{31})r_3 r_1 + \nu_{31}L_{31}\}C_2$$
$$+ \{\mu_{23}r_{23} + 2(|m_2| |m_3| + \mu_{23})r_2 r_3 + \nu_{23}L_{23}\}C_3 = 0.$$

In this equation the linear expressions L_{jk} do not prevent $H = 0$ from representing a circle.

As is easily seen, the results for both subcases agree with our previous ones if the m_j are made real numbers.

2. Zeros of $F - \lambda$. Likewise our methods permit us to secure the locus of the zeros of $F - \lambda$, where λ is an arbitrary constant, real or complex. As

a sample of such results, we state the following theorem for the case of real positive m_j and of finite regions C_j.

The locus of the zeros of $F - \lambda$ is the interior and the boundary of all the ovals of the $(q+1)$-circular $2(q+1)$-ic curve

$$(7.21) \qquad \lambda\bar{\lambda} - \sum_1^{q+1} \frac{m_j L_j}{C_j} + \sum_1^{q+1} \frac{nm_j}{C_j} - \sum_{j=1,k=j+1}^{q+1} \frac{m_j m_k \sigma_{jk}}{C_j C_k} = 0,$$

where $n = \sum_1^{q+1} m_j$, and

$$L_j = \bar{\lambda}(\bar{z} - \bar{\alpha}_j) + \lambda(z - \alpha_j).$$

The singular foci of (6.21) *are the zeros of $F - \lambda$ for $z_j = \alpha_j$.*

As immediate applications of these results, we mention the location of the zeros of linear combinations of either a rational function and its derivative or the product of two binary forms and the jacobian of these forms.[*] For instance, the expression $f'(z) - \lambda f(z)$, where $f(z)$ is a rational function and λ an arbitrary constant, has its zeros at the multiple roots of $f(z)$ and at the zeros of $F - \lambda$, where F is the partial fraction f'/f.

In subsequent papers we hope to study further the properties of the locus R and to give additional specializations and generalizations.

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[*] Cf. Walsh, Bulletin of the American Mathematical Society, vol. 30 (1924), pp. 51–62, for a study of the zeros of linear combinations of a polynomial and k of its derivatives.

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